

Disclination in Lorentz Space-Time

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The disclination in Lorentz space-time is studied in detail by means of topological properties of ϕ -mapping. It is found the space-time disclination can be described in term of a Dirac spinor. The size of the disclination, which is proved to be the difference of two sets of $su(2)$ -like monopoles expressed by two mixed spinors, is quantized topologically in terms of topological invariants—winding number. The projection of space-time disclination density along an antisymmetric tensor field is characterized by Brouwer degree and Hopf index.

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I. INTRODUCTION

The topological defects certainly offer interesting arena for the study of astrophysics and cosmology. These are objects that may have formed during phase transitions in the cooling down of the Early-Universe and may have played a key role in the formation of the large scale structure of the universe mainly through their gravitational interaction [1,2].

As a kind of topological defect, the disclination is caused by inserting a solid angle into the flat space-time, which is described by the Riemannian curvature $R_{\mu\nu\sigma}^\lambda$ or $SO(4)$ ($SO(3,1)$ in Lorentz space-time) gauge field tensor 2-form F^{ab}

$$\theta^{ab} = \frac{1}{2} R_{\mu\nu\sigma}^\lambda e_\lambda^a e^{\sigma b} dx^\mu \wedge dx^\nu = -F^{ab} \quad (1)$$

in which θ^{ab} is the disclination density. The size of the disclination can be represented by the means of the surface integral of the projection of the disclination density along an antisymmetric tensor field ϕ^{ab}

$$\Omega = \oint_\Sigma \theta^{ab} \phi^{ab} = - \oint_\Sigma F^{ab} \phi^{ab} \quad (2)$$

where Σ is a closed surface including the disclinations. The new quantity Ω defined by (2) is dimensionless. In Riemann-Cartan geometry, this effect is showed by the integral of the affine curvature along a closed surface. Duan, Duan and Zhang [3] had discussed the disclinations in deformable material media by applying the gauge field theory and decomposition theory of gauge potential. In their works, the projection of disclination density along the gauge parallel vector was found corresponding to a set of isolated disclinations in the three dimensional sense and being topologically quantized.

In previous publication [4], we studied the topological structure of disclination in the 4-dimensional space-time of Euclidean signature. With suitable coordinates, the projection of disclination density along a gauge parallel tensor was found to be null everywhere except some singular points where it has three dimensional δ -function singularities. In fact, just like the magnetic monopole theory, the topology of disclination projection is described by the $U(1)$ projection of the gauge group. For $so(4) \cong su(2) \otimes su(2)$, the task is turned to find the topology of the subgroup $U(1) \otimes U(1)$.

However, in physics we are more interested in the space-time of Lorentzian signature. There is no way in extending the methods to the case of indefinite metric, because the Lorentz group $SO(3,1)$ is not, like $SO(4)$, decomposable into a product of $SU(2)$ groups. In the present work, we find a proper way to obtain the $U(1)$ projection of the Lorentz group by making using of spinors. δ -function singularities of the projection of the disclination density are found again. The size of the space-time disclination (2) is found to be a difference of two sets of solid angles rather than a sum. The topological structure of the disclination is discussed in detail.

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II. THE GAUGE THEORY OF LORENTZ GROUP AND DISCLINATION PROJECTION

In this paper, we use $a = 1, 2, 3, 4$ to denote the Lorentz script with the fourth component be pure imaginary and the signature of the metric of Lorentz group is $(+++)$.

Let the four-dimensional Dirac matrix γ_a are the bases of Clifford algebra which satisfies

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}. \quad (3)$$

The antisymmetric tensor field ϕ^{ab} on \mathbf{M} can be expressed in the following matrix form

$$\phi = \frac{1}{2} \phi^{ab} I_{ab}.$$

in which I_{ab} are the generators of the group $SO(3, 1)$

$$I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]. \quad (4)$$

The covariant derivative 1-form of ϕ is given by

$$D\phi = d\phi - [\omega, \phi],$$

where ω is the spin connection (gauge potential) which is also Lie algebra-valued

$$\omega = \frac{1}{2} \omega^{ab} I_{ab}. \quad (5)$$

The curvature 2-form (gauge field) is

$$F = d\omega - \omega \wedge \omega. \quad (6)$$

Take the chiral representation

$$\gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

The generators I_{ab} are

$$I_{ij} = \frac{i}{2} \varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad I_{i4} = \frac{i}{2} \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \quad (8)$$

in which $i, j = 1, 2, 3$. In the chiral representation the gauge field is presented as

$$\begin{aligned} F &= \frac{1}{2} F^{ab} I_{ab} \\ &= \frac{i}{2} \begin{pmatrix} \frac{1}{2} F^{ij} \varepsilon_{ijk} \sigma_k + F^{i4} \sigma_i & \\ & \frac{1}{2} F^{ij} \varepsilon_{ijk} \sigma_k - 2F^{i4} \sigma_i \end{pmatrix} \\ &= \begin{pmatrix} F_R & \\ & F_L \end{pmatrix}, \end{aligned} \quad (9)$$

where F_R and F_L are defined as

$$F_R = \frac{i}{2} \left(\frac{1}{2} \varepsilon^{ijk} F^{jk} + F^{i4} \right) \sigma_i \quad F_L = \frac{i}{2} \left(\frac{1}{2} \varepsilon^{ijk} F^{jk} - F^{i4} \right) \sigma_i. \quad (10)$$

Also we can define ω_R and ω_L as

$$\omega_R = \frac{i}{2} \left(\frac{1}{2} \varepsilon^{ijk} \omega^{jk} + \omega^{i4} \right) \sigma_i \quad \omega_L = \frac{i}{2} \left(\frac{1}{2} \varepsilon^{ijk} \omega^{jk} - \omega^{i4} \right) \sigma_i. \quad (11)$$

It can be proved that

$$F_R = d\omega_R - \omega_R \wedge \omega_R \quad F_L = d\omega_L - \omega_L \wedge \omega_L. \quad (12)$$

There exist the relations

$$F_R^\dagger = -F_L \quad \omega_R^\dagger = -\omega_L. \quad (13)$$

A Dirac spinor ψ can be written as

$$\psi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}, \quad (14)$$

where χ_R and χ_L are the right-handed Weyl spinor and left-handed Weyl spinor respectively. The Dirac spinor ψ transforms under $S \in SO(3, 1)$ as

$$\psi \rightarrow S\psi. \quad (15)$$

The covariant derivative of ψ is

$$D\psi = d\psi - \omega\psi. \quad (16)$$

For the Dirac conjugate spinor

$$\bar{\psi} = \psi^\dagger \gamma_4 = (\chi_L^\dagger, \chi_R^\dagger), \quad (17)$$

the covariant derivative is

$$D\bar{\psi} = d\bar{\psi} + \bar{\psi}\omega. \quad (18)$$

Then the covariant derivatives of $\chi_{R,L}$ and $\chi_{R,L}^\dagger$ are

$$D\chi_R = d\chi_R - \omega_R\chi_R \quad D\chi_L = d\chi_L - \omega_L\chi_L \quad (19)$$

$$D\chi_R^\dagger = d\chi_R^\dagger + \chi_R^\dagger\omega_R \quad D\chi_L^\dagger = d\chi_L^\dagger + \chi_L^\dagger\omega_L. \quad (20)$$

Define an antisymmetric tensor ϕ^{ab} as

$$\phi^{ab} = -\frac{i}{2}\bar{\psi}I_{ab}\psi \quad (21)$$

It is easy to prove ϕ^{ab} is a $SO(3, 1)$ covariant tensor with ϕ^{ij} ($i, j = 1, 2, 3$) are real and ϕ^{i4} are pure imaginary. Then the projection of curvature 2-form F^{ab} on ϕ^{ab} is

$$F^{ab}\phi^{ab} = -i\bar{\psi}F\psi \quad (22)$$

Therefore, we get the disclination projection density in Lorentz Space-time as

$$F_p = -F^{ab}\phi^{ab} = i\bar{\psi}F\psi \quad (23)$$

which is a Lorentz invariant.

Noticing that

$$F\psi = -D^2\psi, \quad (24)$$

we get

$$-i\bar{\psi}F\psi = iD(\bar{\psi}D\psi) - iD\bar{\psi} \wedge D\psi. \quad (25)$$

For $\bar{\psi}D\psi$ is a Lorentz scalar, we have

$$\begin{aligned} iD(\bar{\psi}D\psi) &= id(\bar{\psi}D\psi) \\ &= id\bar{\psi} \wedge d\psi - id(\bar{\psi}\omega\psi). \end{aligned} \quad (26)$$

Denote the projection of the connection 1-form ω^{ab} on ϕ^{ab} as

$$A = \omega^{ab}\phi^{ab} = -i\bar{\psi}\omega\psi. \quad (27)$$

Substituting equations (25), (26) and (27) into the disclination projection density (23), we get

$$\begin{aligned} F_p &= -id\bar{\psi} \wedge d\psi - dA + iD\bar{\psi} \wedge D\psi \\ &= -id\chi_L^\dagger \wedge d\chi_R - id\chi_R^\dagger \wedge d\chi_L - dA + iD\bar{\psi} \wedge D\psi. \end{aligned} \quad (28)$$

Define two mixed spinors as

$$\xi_+ = \frac{1}{2}(\chi_R + \chi_L) \quad (29)$$

$$\xi_- = \frac{1}{2}(\chi_R - \chi_L). \quad (30)$$

We get the disclination projection density in terms of the mixed spinors

$$F_p = -2id\xi_+^\dagger \wedge d\xi_+ + 2id\xi_-^\dagger \wedge d\xi_- + dA + iD\bar{\psi} \wedge D\psi. \quad (31)$$

Choose proper ψ such that

$$\psi^\dagger\psi = 4 \quad \text{and} \quad \bar{\psi}\psi = 0,$$

which makes ξ_+ and ξ_- satisfy

$$\xi_+^\dagger \xi_+ = \xi_-^\dagger \xi_- = 1. \quad (32)$$

For the nontrivial topology of space-time, there must exist some points where the mixed spinors ξ_\pm are singular. Therefore F_p is not a total derivative globally. For the mixed spinors do not transform covariantly under the Lorentz transformation, the first term and the second term of the disclination projection density (31) do not remain invariant under the Lorentz transformation respectively. However, the total disclinations projection density F_d do hold invariant.

Define two vectors n_\pm^i as

$$n_\pm^i = \xi_\pm^\dagger \sigma^i \xi_\pm, \quad (33)$$

which make n_\pm^i be unit vectors

$$n_\pm^i n_\pm^i = 1. \quad (34)$$

Substitute (33) into the first and the second terms of the right hand side of equation (31), it can be proved

$$-2id\xi_\pm^\dagger \wedge d\xi_\pm = \frac{1}{2}\varepsilon^{ijk} n_\pm^i dn_\pm^j \wedge dn_\pm^k. \quad (35)$$

The definition (33) actually gives a Hopf map

$$S^3 \rightarrow S^2 \quad (36)$$

and equation (35) gives two $su(2)$ -like monopoles [5,6]. If ψ is taken as a gauge parallel spinor

$$D\psi = 0, \quad (37)$$

the disclination projection density becomes

$$F_p = \frac{1}{2}\varepsilon^{ijk} n_+^i dn_+^j \wedge dn_+^k - \frac{1}{2}\varepsilon^{ijk} n_-^i dn_-^j \wedge dn_-^k + dA. \quad (38)$$

Then the size of the space-time disclination is

$$\Omega = \oint_\Sigma \frac{1}{2}\varepsilon^{ijk} n_+^i dn_+^j \wedge dn_+^k - \frac{1}{2}\varepsilon^{ijk} n_-^i dn_-^j \wedge dn_-^k + dA. \quad (39)$$

For the surface Σ is closed, the third term of the right hand side of (39) contribute nothing to Ω , i.e.

$$\Omega = \oint_{\Sigma} \frac{1}{2} \varepsilon^{ijk} n_+^i dn_+^j \wedge dn_+^k - \frac{1}{2} \varepsilon^{ijk} n_-^i dn_-^j \wedge dn_-^k. \quad (40)$$

Using Stokes formula, Ω can be expressed as

$$\Omega = \int_V \frac{1}{2} \varepsilon^{ijk} dn_+^i \wedge dn_+^j \wedge dn_+^k - \frac{1}{2} \varepsilon^{ijk} dn_-^i \wedge dn_-^j \wedge dn_-^k, \quad (41)$$

in which $\partial V = \Sigma$. Let us choose coordinates $y = (u^1, u^2, u^3, v)$ on \mathbf{M} such that $u = (u^1, u^2, u^3)$ are intrinsic coordinate on V . For the coordinate component v does not belong to V , Ω becomes

$$\begin{aligned} \Omega &= \int_V \left(\frac{1}{2} \varepsilon^{ijk} \partial_\alpha n_+^i \partial_\beta n_+^j \partial_\gamma n_+^k - \frac{1}{2} \varepsilon^{ijk} \partial_\alpha n_-^i \partial_\beta n_-^j \partial_\gamma n_-^k \right) du^\alpha \wedge du^\beta \wedge du^\gamma \\ &= \int_V \left(\frac{1}{2} \varepsilon^{ijk} \varepsilon^{\alpha\beta\gamma} \partial_\alpha n_+^i \partial_\beta n_+^j \partial_\gamma n_+^k - \frac{1}{2} \varepsilon^{ijk} \varepsilon^{\alpha\beta\gamma} \partial_\alpha n_-^i \partial_\beta n_-^j \partial_\gamma n_-^k \right) d^3u, \end{aligned}$$

where $\alpha, \beta, \gamma = 1, 2, 3$ and $\partial_\alpha = \partial/\partial u^\alpha$. Define solid angle densities as

$$\rho = \frac{1}{2} \varepsilon^{ijk} \varepsilon^{\alpha\beta\gamma} \partial_\alpha n_+^i \partial_\beta n_+^j \partial_\gamma n_+^k - \frac{1}{2} \varepsilon^{ijk} \varepsilon^{\alpha\beta\gamma} \partial_\alpha n_-^i \partial_\beta n_-^j \partial_\gamma n_-^k. \quad (42)$$

Then we get

$$\Omega = \int_V \rho d^3u. \quad (43)$$

III. TOPOLOGICAL STRUCTURE OF THE SPACE-TIME DISCLINATION

Now one see the final equation (42) is similar to that of space-time of Euclidean signature in the previous paper [4] except the disclination size is a difference rather than a sum.

The unit vectors n_\pm^i , can be express as

$$n_\pm^i = \frac{\phi_\pm^i}{\|\phi_\pm\|} \quad (44)$$

where ϕ_\pm^i , are smooth vectors along the direction of n_\pm^i respectively. Substituting it into (42), and using the Laplacian relation we can obtain the topological structure of the disclination in Lorentz space-time

$$\rho = 4\pi \delta^3(\phi_+) J\left(\frac{\phi_+}{u}\right) - 4\pi \delta^3(\phi_-) J\left(\frac{\phi_-}{u}\right) \quad (45)$$

and

$$\Omega = 4\pi \int_V \left(\delta^3(\phi_+) J\left(\frac{\phi_+}{u}\right) - \delta^3(\phi_-) J\left(\frac{\phi_-}{u}\right) \right) d^3u. \quad (46)$$

in which the Jacobian $J(\frac{\phi_\pm}{u})$ are

$$\varepsilon^{ijk} J\left(\frac{\phi_\pm}{u}\right) = \varepsilon^{\alpha\beta\gamma} \partial_\alpha \phi_\pm^i \partial_\beta \phi_\pm^j \partial_\gamma \phi_\pm^k. \quad (47)$$

Suppose that $\phi_\pm^i(x)$, possess K_\pm isolated zeros respectively, the solutions of $\phi_\pm(u^1, u^2, u^3, v) = 0$ can be expressed in terms of $u = (u^1, u^2, u^3)$ as

$$u^i = z^i(v), \quad i = 1, 2, 3 \quad (48)$$

and

$$\phi^i(z_l^1(v), z_l^2(v), z_l^3(v), v) \equiv 0, \quad (49)$$

where the subscript $l = 1, 2, \dots, K$ represents the l th zero of ϕ^i , i.e.

$$\phi_{\pm}^i(z_l^i) = 0, \quad l = 1, 2, \dots, K_{\pm}; \quad A = 1, 2, 3. \quad (50)$$

It is easy to get the following formula from the ordinary theory of δ -function that

$$\delta^3(\phi_{\pm})J\left(\frac{\phi_{\pm}}{u}\right) = \sum_{i=1}^{K_{\pm}} \beta_{\pm i} \eta_{\pm i} \delta^3(u - z_{\pm i}), \quad (51)$$

in which

$$\eta_{\pm i} = \text{sign}J\left(\frac{\phi_{\pm}}{u}\right)|_{x=z_{\pm i}} = \pm 1 \quad (52)$$

is the Brouwer degree of ϕ -mapping and $\beta_{\pm i}$ are positive integers called the Hopf index of map ϕ_{\pm} which means while the point x covers the region neighboring the zero $x = z_{\pm i}$ once, ϕ_{\pm} covers the corresponding region $\beta_{\pm i}$ times. Therefore the slid angle density becomes

$$\rho = 4\pi \sum_{l=1}^{K_+} \beta_{+l} \eta_{+l} \delta^3(u - z_{+l}) - 4\pi \sum_{l=1}^{K_-} \beta_{-l} \eta_{-l} \delta^3(u - z_{-l}) \quad (53)$$

and

$$\Omega = 4\pi \sum_{l=1}^{K_+} \beta_{+l} \eta_{+l} - 4\pi \sum_{l=1}^{K_-} \beta_{-l} \eta_{-l}. \quad (54)$$

It is obvious from (54) that the size of the space-time disclination is quantized for topological reason.

In fact the winding numbers $W_{\pm l}$ of the map ϕ_{\pm} around the zeroes $z_{\pm l}$ are

$$W_{\pm l} = \beta_{\pm l} \eta_{\pm l}$$

Hence, the space-time disclinations is quantized by the winding numbers as

$$\Omega = 4\pi \sum_{l=1}^{K_+} W_{+l} - 4\pi \sum_{l=1}^{K_-} W_{-l}. \quad (55)$$

or by the winding numbers of the map ϕ_{\pm} around the surface Σ

$$\Omega = 4\pi W_+ - 4\pi W_-. \quad (56)$$

The winding number W_{\pm} of the surface Σ can be interpreted or, indeed, defined as the degree of the mappings ϕ_{\pm} onto Σ . Then the space-time disclination is

$$\Omega = 4\pi \deg \phi_+ - 4\pi \deg \phi_-. \quad (57)$$

We find that (53) is the exact density of a system of K_+ and K_- classical point-like objects with quantized “charge” $\beta_{+l} \eta_{+l}$ and $\beta_{-l} \eta_{-l}$ in space-time, i.e. the topological structure of disclinations formally corresponds to a point-like system. These point objects may be called disclination points as in nematic crystals [7]. In [8], it was shown that the existence of disclination points is related to a kind of broken symmetries. The dislocations and disclinations appear as singularities of distortions of an order parameter [9]. In our paper, the disclination points are identified with the isolated zero points of vector field $\phi_{\pm}^i(x)$. From (45) we know that these singularities are those of the disclination density as well.

IV. CONCLUSION

In this paper, we have studied the topological structure, global and local properties of disclinations in the general 4-dimensional Lorentz space-time. By defining an antisymmetric tensor in terms of a Dirac spinor, we get the disclination projection density. The projection of disclination is proved to be the difference of two sets of isolated

disclinations, each of which corresponds to a $su(2)$ -like monopole expressed by some mixed spinor. We showed that the size of space-time disclination is quantized topologically. The positions of the disclinations are determined by the zeroes of two mixed spinors. And the Hopf index and Brouwer degree classify the disclinations and characterize the local nature of the space-time disclinations. For this quantization of the size of the space-time disclinations is related to the space-time curvature directly, it has close relationship with the quantization of space-time.

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